

PROJECTION DECOMPOSITION IN MULTIPLIER ALGEBRAS

VICTOR KAFTAL, P. W. NG, AND SHUANG ZHANG

ABSTRACT. In this paper we present new structural information about the multiplier algebra $\mathcal{M}(\mathcal{A})$ of a σ -unital purely infinite simple C^* -algebra \mathcal{A} , by characterizing the positive elements $A \in \mathcal{M}(\mathcal{A})$ that are strict sums of projections belonging to \mathcal{A} . If $A \notin \mathcal{A}$ and A itself is not a projection, then the necessary and sufficient condition for A to be a strict sum of projections belonging to \mathcal{A} is that $\|A\| > 1$ and that the essential norm $\|A\|_{ess} \geq 1$.

Based on a generalization of the Perera-Rordam weak divisibility of separable simple C^* -algebras of real rank zero to all σ -unital simple C^* -algebras of real rank zero, we show that every positive element of \mathcal{A} with norm greater than 1 can be approximated by finite sums of projections. Based on block tri-diagonal approximations, we decompose any positive element $A \in \mathcal{M}(\mathcal{A})$ with $\|A\| > 1$ and $\|A\|_{ess} \geq 1$ into a strictly converging sum of positive elements in \mathcal{A} with norm greater than 1.

1. Introduction and the main result

In [9] Fillmore raised the following question: Which positive bounded operators on a separable Hilbert space \mathcal{H} can be written as (finite) sums of projections? Fillmore obtained a characterization of the finite rank operators that are sums of projections (see [9] Theorem 1) and of the bounded operators that are the sums of two projections (see [9] Theorem 2).

For infinite sums of projections with convergence in the strong operator topology, this question arose naturally from work on frame theory by Dykema, Freeman, Kornelson, Larson, Ordower and Weber (see [5]). They proved that a sufficient condition for a positive bounded operator $A \in \mathbb{B}(\mathcal{H})$ to be a (possibly infinite) sum of projections converging in the strong operator topology is that its essential norm $\|A\|_{ess}$ is greater than 1 (see [5] Theorem 2). This result served as a basis for further work by Kornelson and Larson [13] and then by Antezana, Massey, Ruiz and Stojanoff [1] on decompositions of positive operators into strongly converging sums of rank one positive operators with preassigned norms. In [12], the necessary and sufficient condition for a positive bounded operator to be a strongly converging sum of projections was obtained by the three authors of this article for the $\mathbb{B}(\mathcal{H})$ case and for the case of a countably decomposable type III von Neumann factor, and for the “diagonalizable” case of type II von Neumann factors.

In this paper, we extend the characterization of the positive operators that are sums of projections to the case of bounded module maps (with adjoints defined) on Hilbert C^* -modules, namely, $\mathbb{B}(\mathcal{H})$ is replaced by the multiplier algebra $\mathcal{M}(\mathcal{A})$ of \mathcal{A} . Dealing with multiplier algebras, we replace the strong operator topology by the strict topology. We point out that when \mathcal{A} is reduced to the algebra of complex numbers \mathbb{C} , then $\mathbb{B}(\mathcal{H}) = \mathcal{M}(\mathcal{K})$, the multiplier algebra of the C^* -algebra \mathcal{K} of compact operators on a separable Hilbert space, and the $*$ -strong operator topology on $\mathbb{B}(\mathcal{H})$ is precisely the strict topology of $\mathcal{M}(\mathcal{K})$.

In this article we generalize the main result of [5] to certain multiplier algebras, stated as follows.

Theorem 1.1. *Let \mathcal{A} be a σ -unital simple purely infinite C^* -algebra and A be a positive element of $\mathcal{M}(\mathcal{A})$. Then A is a strictly converging sum of projections belonging to \mathcal{A} if and only if one of the following mutually exclusive conditions hold:*

- (i) $\|A\|_{ess} > 1$.
- (ii) $\|A\|_{ess} = 1$ and $\|A\| > 1$.
- (iii) $A \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$ is a projection.
- (iv) A is the sum of finitely many projections belonging to \mathcal{A} .

When \mathcal{A} is unital and hence $\mathcal{M}(\mathcal{A}) = \mathcal{A}$, if a positive element $A \in \mathcal{A}$ is a strictly converging sum of (nonzero) projections belonging to \mathcal{A} , then the sum must be finite (Proposition 3.1), stated as the case (iv).

The non-trivial case is thus when $A \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$ where \mathcal{A} is σ -unital but non-unital. Notice that such a C^* -algebra (σ -unital but non-unital simple purely infinite) is necessarily stable (see [32] and [22]) and has real rank zero ([29, 1.2]).

The necessity of the conditions (i)–(iii) is given by Corollary 3.3. The sufficiency of (iii) being trivial, the main focus of this paper is to prove the sufficiency of (i) and (ii).

The proof is arranged in the following way.

In section 2 we prove that all non-elementary, σ -unital, simple C^* -algebras of real rank zero are weakly divisible in the sense of Perera-Rordam in [19], thus generalizing the previous result of [19] from the separable category to the σ -unital category. This weak divisibility property and Fillmore's characterization of the finite rank operators in $\mathbb{B}(\mathcal{H})$ enable us to approximate a positive element with a norm greater than 1 by finite sums of projections (Lemma 2.5.)

In section 3 we prove that a positive element A of $\mathcal{M}(\mathcal{A})$ with essential norm $\|A\|_{ess} > 1$ can be written as a strict sum of projections in \mathcal{A} .

Section 4 deals with the crucial case when $\|A\|_{ess} = 1$ and $\|A\| > 1$. We employ a block tri-diagonal approximation and operator theory techniques to construct a strictly converging sequence of projections $f_k \in \mathcal{A}$ for which $\|f_k A f_k\| > 1$ for all k (Lemma 4.4). From that, we decompose A into a strict sum of projections (Proposition 4.6) and conclude the proof.

Aside from the works on the $\mathbb{B}(\mathcal{H})$ and von Neumann factors cases that have been mentioned above, this paper employs some previous results and ideas on the structures of multiplier algebras of simple purely infinite C^* -algebras scattered in the several papers such as [14], [16], [17], [19], and [21] – [33].

The first and second named authors participated in the NSF supported Workshop in Analysis and Probability, Texas A & M University, Summer 2006, where they first heard from David Larson about the results in [5] and [13] that stimulated this project.

The first and third author were partially supported by grants from the Charles Phelps Taft Research Center.

2. Weak divisibility of σ -unital C^* -algebras of real rank zero

In this section we show that in a σ -unital simple purely infinite C^* -algebra \mathcal{A} , every positive element with norm greater than 1 can be approximated *from underneath* by finite sums of projections. To do so we first extend to all non-elementary

σ -unital C^* -algebras of real rank zero the property of weak divisibility obtained for separable non-elementary simple C^* -algebras of real rank zero by Perera and Rordam in [19, 5.3]. Recall that a C^* -algebra is called non-elementary if it is neither \mathcal{K} nor \mathbb{M}_n for any n . A C^* -algebra is called to be σ -unital, if it has a strictly positive element b , namely, $(bA)^- = (Ab)^- = A$. A C^* -algebra \mathcal{B} is weakly divisible ([19, 5.1, 5.2]) if and only if for any nonzero projection p of \mathcal{B} there exists a unital $*$ -homomorphism from $\mathbb{M}_2 \oplus \mathbb{M}_3$ to $p\mathcal{B}p$.

Proposition 2.1. *If \mathcal{A} is a non-elementary σ -unital simple C^* -algebra of real rank zero, then \mathcal{A} is weakly divisible.*

Proof. By [19, Lemma 5.2] it suffices to show that for each nonzero projection p of \mathcal{A} there exists a unital $*$ -homomorphism from $\mathbb{M}_2 \oplus \mathbb{M}_3$ into $p\mathcal{A}p$. To prove this, we use the result of divisibility of all projections in any simple C^* -algebra of real rank zero in [26, 1.1]: For each pair of projections (q, r) in \mathcal{A} and each natural number n the projection q can be rewritten as a direct sum of mutually orthogonal subprojections

$$q = p_1 \oplus p_2 \oplus \cdots \oplus p_{2^n} \oplus r_0$$

such that p_i is equivalent to p_j for all pairs (i, j) in the sense of Murray-von Neumann and r_0 is equivalent to a subprojection of r .

Applying this result to the case $q = r = p$ and $n = 1$, one has

$$p = p_1 \oplus p_2 \oplus r_0$$

where p_1 is equivalent to p_2 and r_0 is equivalent to a subprojection of p_1 , say r_1 . Choose a partial isometry v such that $p_1 = vv^*$, $p_2 = v^*v$, and set $r_2 = v^*r_1v$. Then

$$p = (p_1 - r_1) \oplus (p_2 - r_2) \oplus r_0 \oplus r_1 \oplus r_2.$$

Then $p_1 - r_1$ and $p_2 - r_2$ are equivalent, and so are r_0, r_1 and r_2 . This decomposition of p into these five projections leads to a unital $*$ -homomorphism from $\mathbb{M}_2 \oplus \mathbb{M}_3$ into $p\mathcal{A}p$. \square

The same idea above also proves the following lemma that will be used as one of the technical ingredients in this article.

Lemma 2.2. *Let \mathcal{A} be a non-elementary σ -unital simple C^* -algebra of real rank zero. Then for every integer $n \geq 1$ and for every nonzero projection p of \mathcal{A} there exists a unital $*$ -embedding of $\mathbb{M}_{2^n} \oplus \mathbb{M}_{2^n+1}$ into $p\mathcal{A}p$.*

Proof. Applying [26, 1.1] to the case $q = r = p$ and arbitrary natural number n , one has

$$p = p_1 \oplus p_2 \oplus \cdots \oplus p_{2^n} \oplus r_0$$

where p_i is equivalent to p_j for all pair (i, j) and r_0 is equivalent to a subprojection of p_1 , say r_1 . For each k choose a partial isometry $v_k \in p\mathcal{A}p$ such that $p_1 = v_k v_k^*$ and $p_k = v_k^* v_k$. Let $r_k = v_k^* r_1 v_k$. Then

$$p = (p_1 - r_1) \oplus (p_2 - r_2) \oplus \cdots \oplus (p_{2^n} - r_{2^n}) \oplus r_0 \oplus r_1 \oplus r_2 \oplus \cdots \oplus r_{2^n}.$$

Then $p_i - r_i$ and $p_j - r_j$ are equivalent for all pairs (i, j) , and so are r_0, r_1, \dots, r_{2^n} . This decomposition of p leads to a unital $*$ -homomorphism from $\mathbb{M}_{2^n} \oplus \mathbb{M}_{2^n+1}$ into $p\mathcal{A}p$. \square

We need the following approximation property for positive elements in a C^* -algebra of real rank zero.

Lemma 2.3. *Let \mathcal{C} be a C^* -algebra of real rank zero and c be any positive element in \mathcal{C} . For $\epsilon > 0$ there exist pairwise orthogonal projections p_1, p_2, \dots, p_n in \mathcal{C} and positive real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ such that*

- (i) $\|\alpha_1 p_1 + \alpha_2 p_2 + \dots + \alpha_n p_n - c\| < \epsilon$
- (ii) $\alpha_1 p_1 + \alpha_2 p_2 + \dots + \alpha_n p_n \leq c$.

Proof. Without loss of generality, assume that $\epsilon < 2\|c\|$. Let g be the piecewise linear function

$$g(x) =: \begin{cases} 0 & 0 \leq x \leq \epsilon/2 \\ x - \epsilon/2 & x > \epsilon/2. \end{cases}$$

The hereditary subalgebra of \mathcal{C} generated by $\overline{g(c)\mathcal{C}g(c)}$ still has real rank zero ([3]). Thus one can find a positive element $d \in \overline{g(c)\mathcal{C}g(c)}$ with finite spectrum, say $d = \alpha_1 p_1 + \alpha_2 p_2 + \dots + \alpha_n p_n$, such that $\|g(c) - d\| < \epsilon/2$. It follows that

$$\|c - \alpha_1 p_1 + \alpha_2 p_2 + \dots + \alpha_n p_n\| \leq \|c - g(c)\| + \|g(c) - d\| < \epsilon/2 + \epsilon/2 = \epsilon.$$

The key point is to prove that $d \leq c$. Assume without loss of generality that \mathcal{C} act faithfully and non-degenerately on a Hilbert space \mathcal{H} . Let $q =: \chi_{[\epsilon/2, \infty)}(c)$. For $\xi \in q\mathcal{H}$, one has that

$$\begin{aligned} \langle (c - d)\xi, \xi \rangle &= \langle (c - g(c))\xi, \xi \rangle + \langle (g(c) - d)\xi, \xi \rangle \\ &= \epsilon \langle \xi, \xi \rangle + \langle (g(c) - d)\xi, \xi \rangle \\ &\geq (\epsilon - \|g(c) - d\|) \langle \xi, \xi \rangle \\ &\geq 0. \end{aligned}$$

If $\xi \in q^\perp \mathcal{H}$, then $d\xi = 0$ because $d \in \overline{g(c)\mathcal{C}g(c)}$, and hence also $\langle (c - d)\xi, \xi \rangle = \langle c\xi, \xi \rangle \geq 0$. Therefore, $c \geq d$, as wanted. \square

We will use the following result due to Fillmore [9, Thm. 1] (see also [5, Prop. 6] and [12, 2.5, 2.6].)

Proposition 2.4. *Let tr be the natural (non-normalized) trace on the algebra \mathbb{M}_n of n by n complex matrices and $A \in \mathbb{M}_n$ be a positive matrix. Then A is a sum of projections in \mathbb{M}_n if and only if $tr(A)$ is an integer and $tr(A) \geq \text{rank}(A)$.*

Recall that all simple purely infinite C^* -algebras have real rank zero ([29, 1.2]). The following lemma is one of the two central technical ingredients of this article.

Lemma 2.5. *Let \mathcal{A} be a σ -unital purely infinite simple C^* -algebra and $A \in \mathcal{A}$ be a positive element with $\|A\| > 1$. Then for every $\epsilon > 0$ there exist positive elements $A_1, A_2 \in \mathcal{A}$ such that*

- (i) $A = A_1 + A_2$,
- (ii) A_1 is the sum of finitely many projections belonging to \mathcal{A} , and
- (iii) $\|A_2\| < \epsilon$.

Proof. By Lemma 2.3 we can assume without loss of generality that A is a positive element with finite spectrum and with norm strictly greater than one. Then there are nonzero pairwise orthogonal projections

$$e_1, e_2, \dots, e_m, f_1, f_2, \dots, f_n \in \mathcal{A}$$

and strictly positive real numbers $\lambda_1, \lambda_2, \dots, \lambda_m, \mu_1, \mu_2, \dots, \mu_n$ such that

$$A = \sum_{i=1}^m \lambda_i e_i + \sum_{j=1}^n \mu_j f_j,$$

where $1 < \lambda_i$ and $0 < \mu_j \leq 1$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Note that $\|A\| > 1$ implies $m \geq 1$; but $n = 0$ is possible.

Choose N large enough in the form 2^k such that there are positive integers k_i, k'_i, l_j, l'_j for all $1 \leq i \leq m$ and $1 \leq j \leq n$ satisfying the following inequalities:

$$\begin{aligned} 1 < k_i/N < \lambda_i \quad \text{and} \quad 1 < k'_i/(N+1) < \lambda_i, \\ l_j/N < \mu_j \quad \text{and} \quad l'_j/(N+1) < \mu_j, \\ 0 < \lambda_i - k_i/N < \frac{\epsilon}{2} \quad \text{and} \quad 0 < \lambda_i - k'_i/(N+1) < \frac{\epsilon}{2}, \\ 0 < \mu_j - l_j/N < \frac{\epsilon}{2} \quad \text{and} \quad 0 < \mu_j - l'_j/(N+1) < \frac{\epsilon}{2}. \end{aligned}$$

By Lemma 2.2 there exists for $1 \leq i \leq m$ a unital $*$ -homomorphism from $\mathbb{M}_N \oplus \mathbb{M}_{N+1}$ onto a C^* -subalgebra \mathcal{B}_i of the corner $e_i \mathcal{A} e_i$, and for $1 \leq j \leq n$ there is a unital $*$ -homomorphism from $\mathbb{M}_N \oplus \mathbb{M}_{N+1}$ onto a C^* -subalgebra \mathcal{C}_j of $f_j \mathcal{A} f_j$. Notice that for a given i , the projections in \mathcal{B}_i that correspond to the minimal projections of \mathbb{M}_N are all mutually equivalent, but in general they are not comparable to the minimal projections in \mathcal{C}_j or in $\mathcal{B}_{i'}$ for $i \neq i'$ or to those in \mathcal{B}_i that correspond to the minimal projections of \mathbb{M}_{N+1} . The identity of \mathcal{B}_i is e_i and the identity of \mathcal{C}_j is f_j and this way, each summand $\lambda_i e_i$ is identified with a direct sum of two diagonal matrices, say $B_i = B_{i1} \oplus B_{i2}$ in \mathcal{B}_i , where B_{i1} is a matrix of size $N \times N$, B_{i2} is a matrix of size $(N+1) \times (N+1)$, and both have all diagonal entries λ_i . Similarly, each summand $\mu_j f_j$ is identified with a direct sum of two diagonal matrices, say $C_j = C_{j1} \oplus C_{j2}$ in \mathcal{C}_j , where C_{j1} is of size $N \times N$, C_{j2} is of size $(N+1) \times (N+1)$, and both have all diagonal entries μ_j .

Modify $B_i = B_{i1} \oplus B_{i2}$ to $B'_i = B'_{i1} \oplus B'_{i2}$ where B'_{i1} has the same matrix units as B_{i1} but has all diagonal entries k_i/N instead of λ_i and B'_{i2} has the same matrix units as B_{i2} but has all diagonal entries $k'_i/(N+1)$ instead of λ_i . Similarly, modify $C_j = C_{j1} \oplus C_{j2}$ to $C'_j = C'_{j1} \oplus C'_{j2}$ by replacing the diagonal entries μ_j of C_{j1} with l_j/N and the diagonal entries μ_j of C_{j2} with $l'_j/(N+1)$. Let

$$A' = \sum_{i=1}^m B'_i + \sum_{j=1}^n C'_j.$$

Notice that all the matrices B'_{i1} and C'_{j1} have rank N and all the matrices B'_{i2} and C'_{j2} have rank $N+1$. The conditions defining k_j, k'_j, l_j, l'_j imply:

$$\begin{aligned} 0 \leq A' \leq A \quad \text{and} \quad \|A - A'\| < \frac{\epsilon}{2}, \\ \text{tr}(B'_{i1}) = N(k_i/N) = k_i \quad \text{and} \quad \text{tr}(B'_{i2}) = (N+1)k'_i/(N+1) = k'_i, \\ \text{tr}(C'_{j1}) = N(l_j/N) = l_j \quad \text{and} \quad \text{tr}(C'_{j2}) = (N+1)l'_j/(N+1) = l'_j. \end{aligned}$$

Since

$$\text{tr}(B'_{i1}) = k_i > N = \text{rank}(B'_{i1}) \quad \text{and} \quad \text{tr}(B'_{i2}) = k'_i > N+1 = \text{rank}(B'_{i2}),$$

by Proposition 2.4 each B'_{i1} and B'_{i2} is a sum of projections. If $n = 0$, $A' = \sum_{i=1}^m B'_{i1} \oplus B'_{i2}$ is a sum of projections and then setting $A_1 = A'$ and $A_2 = A - A'$ will satisfy the thesis.

From now on assume $n \geq 1$. Since $\sum_{i=2}^m B'_{i1} \oplus B'_{i2}$ is a sum of projections, it is enough to prove that $B'_1 + \sum_{j=1}^n C'_j$ is also a sum of projections. Let e_{11} (resp., e_{12}) be the identity of the copy of \mathbb{M}_N (resp., \mathbb{M}_{N+1}) in \mathcal{B}_1 . Then $B'_1 = \frac{k_1}{N}e_{11} + \frac{k'_1}{N+1}e_{12}$. Since $1 < \frac{k_1}{N}$, there exists for each $1 \leq j \leq n$ an integer multiple of N , say L_j , such that

$$L_j \left(\frac{k_1}{N} - 1 \right) \geq N - l_j.$$

For every $1 \leq j \leq n$, let f_{j1} be a minimal projection of C'_{j1} . Denote by $N \cdot f_{j1}$ the identity of the copy of \mathbb{M}_N in \mathcal{C}_j . Then $C'_{j1} = \frac{l_j}{N}(N \cdot f_{j1})$. Since the corner $e_{11}\mathcal{A}e_{11}$ of \mathcal{A} is still simple and purely infinite, one can recursively find $\sum_{j=1}^n L_j$ mutually orthogonal projections in $e_{11}\mathcal{A}e_{11}$, where for each $1 \leq j \leq n$, L_j of these projections are equivalent to f_{j1} and we denote their sum by $L_j \cdot f_{j1}$. Then

$$D := \sum_{j=1}^n \frac{k_1}{N} (L_j \cdot f_{j1}) + \sum_{j=1}^n C'_{j1} = \sum_{j=1}^n \left(\frac{k_1}{N} (L_j \cdot f_{j1}) + \frac{l_j}{N} (N \cdot f_{j1}) \right)$$

and for each j , $\frac{k_1}{N}(L_j \cdot f_{j1}) + \frac{l_j}{N}(N \cdot f_{j1})$ is a matrix of size $L_j + N$ and trace $\frac{L_j k_1}{N} + l_j \geq L_j + N$ and hence is the sum of projections by Proposition 2.4.

Similarly, there exists for each $1 \leq j \leq n$ an integer multiple of $N + 1$, say L'_j , such that

$$L'_j \left(\frac{k_1}{N+1} - 1 \right) \geq N + 1 - l'_j.$$

For every $1 \leq j \leq n$, let f_{j2} be a minimal projection of C'_{j2} , $(N + 1) \cdot f_{j2}$ the identity of the copy of \mathbb{M}_{N+1} in \mathcal{C}_j . $L'_j \cdot f_{j2}$ the sum of orthogonal subprojections of e_{12} equivalent to f_{j2} so that $L'_j \cdot f_{j2}$ are also mutually orthogonal. Then

$$D' := \sum_{j=1}^n \frac{k_1}{N+1} (L'_j \cdot f_{j2}) + \sum_{j=1}^n C'_{j2} = \sum_{j=1}^n \left(\frac{k_1}{N+1} (L'_j \cdot f_{j2}) + \frac{l'_j}{N+1} ((N+1) \cdot f_{j2}) \right)$$

is the sum of projections by the same argument as for D .

Finally, let $e' = e_1 - \sum_{j=1}^n (L_j \cdot f_{j1}) - \sum_{j=1}^n (L'_j \cdot f_{j2})$. Then

$$B'_1 + \sum_{j=1}^n C'_j = D + D' + \frac{k_1}{N} e'.$$

If $e' \neq 0$, by the same argument as for the case $n = 0$ one can find a sum of projections D'' for which $\|D'' - \frac{k_1}{N} e'\| < \frac{\epsilon}{2}$. Then setting

$$A_1 := \sum_{i=2}^m B'_{i1} \oplus B'_{i2} + D + D' + D''$$

and $A_2 := A - A_1$ satisfies the thesis. \square

3. The cases $A \in \mathcal{A}$ and $\|A\|_{ess} > 1$

We first discuss when a positive operator A in a σ -unital simple purely infinite C^* -algebra \mathcal{A} is a strict sum of projections in \mathcal{A} .

Proposition 3.1. *Let \mathcal{A} be a σ -unital C^* -algebra with an approximate identity of projections and A be a positive element in \mathcal{A} . If A is the strict sum of projections belonging to \mathcal{A} , then A must be the sum of finitely many projections belonging to \mathcal{A} .*

Proof. We will reason by contradiction. Assume that $\{p_k\}_{k=1}^\infty$ is an infinite sequence of nonzero projections in \mathcal{A} such that $A = \sum_{k=1}^\infty p_k$, where the sum converges in the strict topology in $\mathcal{M}(\mathcal{A})$.

Let $\{e_n\}_{n=1}^\infty$ be an approximate unit for \mathcal{A} consisting of an increasing sequence of projections. Note that such an increasing approximate identity of projections indeed exists in \mathcal{A} ([28]). Choose an integer $N \geq 1$ such that for all $n \geq N$, $\|A - e_n A\| < 1/2$. As a consequence,

$$\|(1 - e_N)A(1 - e_N)\| = \|(A - e_N A)(1 - e_N)\| \leq \|A - e_N A\| < 1/2$$

Recall a classical result (for example, see [4, Lemma III.3.1]) that for every $0 < \epsilon < 1$ there exists a $\delta > 0$ such that $p \in \mathcal{A}$ with $\text{dist}(p, (1 - e_N)\mathcal{A}(1 - e_N)) < \delta$ implies the existence of a projection $q \in (1 - e_N)\mathcal{A}(1 - e_N)$ satisfying $\|p - q\| < \epsilon$. Such a projection q is equivalent to p in \mathcal{A} . For $\epsilon = 1/2$ there exists $\delta > 0$. Since $\sum_{k=1}^\infty p_k$ converges in the strict topology on $\mathcal{M}(\mathcal{A})$, let $K \geq 1$ be such that $\|p_k e_N\| < \delta/3$ for all $k \geq K$. Hence, for all $k \geq K$,

$$\|(1 - e_N)p_k(1 - e_N) - p_k\| \leq \|-p_k e_N - e_N p_k + e_N p_k e_N\| \leq 3\|p_k e_N\| < \delta$$

Thus $\text{dist}(p_K, (1 - e_N)\mathcal{A}(1 - e_N)) < \delta$. It follows from the classical result stated above that there is a projection

$$q \in (1 - e_N)\mathcal{A}(1 - e_N) \quad \text{with} \quad \|p_K - q\| < 1/2.$$

Now let $B = \sum_{k \neq K} p_k + q$. Then

$$\|B - A\| = \|q - p_K\| < 1/2,$$

and hence,

$$\|(1 - e_N)B(1 - e_N) - (1 - e_N)A(1 - e_N)\| < 1/2.$$

Applying the triangle inequality, one has

$$\|(1 - e_N)B(1 - e_N)\| < 1/2 + \|(1 - e_N)A(1 - e_N)\| < 1.$$

On the other hand, $(1 - e_N)B(1 - e_N) \geq q$ implies $\|(1 - e_N)B(1 - e_N)\| \geq 1$, a contradiction. Therefore, A , as the strict sum of projections, must be a finite sum. \square

We now turn to handle the sufficient condition $\|A\|_{ess} > 1$. Let us first review some elementary facts about the essential norm, which are formulated only for the special cases that we will work with. Let \mathcal{A} be a non-unital C^* -algebra, let π be the canonical homomorphism from $\mathcal{M}(\mathcal{A})$ onto the corona algebra $\mathcal{M}(\mathcal{A})/\mathcal{A}$, and for every $A \in \mathcal{M}(\mathcal{A})$, let $\|A\|_{ess} := \|\pi(A)\|$ denote the essential norm.

Lemma 3.2. *Let \mathcal{A} be a non-unital C^* -algebra.*

(i) *For every positive $A \in \mathcal{M}(\mathcal{A})$,*

$$\|A\|_{ess} = \inf\{\|A(I - a)\| \mid a \in \mathcal{A}^+, \|a\| \leq 1\}.$$

(ii) Let $A \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$ be a positive element, and let a_n be a monotone increasing sequence of positive elements of \mathcal{A} converging to A in the strict topology. Then

$$\|A\|_{ess} = \inf_n \|A - a_n\|.$$

Proof.

(i) Since $Aa \in \mathcal{A}$ for every $a \in \mathcal{A}$, it follows that $\|A\|_{ess} = \|A(I-a)\|_{ess} \leq \|A(I-a)\|$ and hence

$$\|A\|_{ess} \leq \inf\{\|A(I-a)\| \mid a \in \mathcal{A}^+, \|a\| \leq 1\}.$$

If $\|A\|_{ess} = \|A\|$, then the reverse inequality holds by choosing $a = 0$, so assume that $\|A\| > \|A\|_{ess}$. Let $0 < \epsilon < \|A\| - \|A\|_{ess}$, let h be the positive continuous function on the interval $[0, \|A\|]$ defined as

$$h(t) := \begin{cases} 0 & t \in [0, \|A\|_{ess}] \\ \text{linear} & t \in [\|A\|_{ess}, \|A\|_{ess} + \epsilon] \\ 1 & t \in [\|A\|_{ess} + \epsilon, \|A\|] \end{cases},$$

and let $a := h(A)$. Clearly, $a \geq 0$ and $\|a\| = 1$. Via the Gelfand's transformation, identify $C^*(\pi(A))$ with the algebra of complex-valued continuous functions $C(\sigma_e(A))$ defined on the essential spectrum $\sigma_e(A)$ of A . Since h vanishes on $\sigma_e(A)$ and $h \circ \pi = \pi \circ h$, it follows that $\pi(h(A)) = 0$ and hence $h(A) \in \mathcal{A}$. Moreover,

$$\|A(I-a)\| = \|t(1-h(t))\|_\infty \leq \|A\|_{ess} + \epsilon,$$

whence

$$\inf\{\|A(I-a)\| \mid a \in \mathcal{A}^+, \|a\| \leq 1\} \leq \|A\|_{ess}.$$

Thus equality holds, proving (i).

(ii) Since for every n

$$\|A\|_{ess} = \|A - a_n\|_{ess} \leq \|A - a_n\|,$$

it follows that

$$\|A\|_{ess} \leq \inf_n \|A - a_n\|.$$

For every positive contraction $a \in \mathcal{A}$ and every n

$$\|A - a_n\|^{1/2} = \|(A - a_n)^{1/2}\| \leq \|(A - a_n)^{1/2}a\| + \|(A - a_n)^{1/2}(I - a)\|.$$

Since $0 \leq A - a_n \leq A$,

$$\|(A - a_n)^{1/2}(I - a)\|^2 = \|(I - a)(A - a_n)(I - a)\| \leq \|(I - a)A(I - a)\| = \|A^{1/2}(I - a)\|^2.$$

But then

$$\|A^{1/2}(I - a)\| \geq \|(A - a_n)^{1/2}(I - a)\| \geq \|A - a_n\|^{1/2} - \|(A - a_n)^{1/2}a\|.$$

Since $A - a_n \rightarrow 0$ in the strict topology it follows that $\|(A - a_n)^{1/2}a\| \rightarrow 0$. Since a_n is monotone increasing, it follows that $\|A - a_n\|^{1/2} \rightarrow \inf_n \|A - a_n\|^{1/2}$ and hence

$$\|A^{1/2}(I - a)\| \geq \inf_n \|A - a_n\|^{1/2}.$$

Thus

$$\inf\{\|A^{1/2}(I - a)\| \mid a \in \mathcal{A}^+, \|a\| \leq 1\} \geq \inf_n \|A - a_n\|^{1/2}$$

and by (i),

$$\|A^{1/2}\|_{ess} \geq \inf_n \|A - a_n\|^{1/2}.$$

Since $\|A\|_{ess} = \|A^{1/2}\|_{ess}^2$, it follows that

$$\|A\|_{ess} \geq \inf_n \|A - a_n\|,$$

which concludes the proof. \square

Corollary 3.3. *Let \mathcal{A} be a non-unital C^* -algebra and let $A = \sum_{j=i}^{\infty} a_j$ where $a_j \in \mathcal{A}^+$, $\|a_j\| \geq 1$ for all j and the series converges in the strict topology of $\mathcal{M}(\mathcal{A})$. Then $\|A\|_{ess} \geq 1$.*

Every σ -unital C^* -algebra \mathcal{A} has a strictly positive element $b \in \mathcal{A}$, i.e., a positive element for which $(b\mathcal{A})^- = (\mathcal{A}b)^- = \mathcal{A}$. As usual, one can assume that $\|b\| = 1$. Define a seminorm on $\mathcal{M}(\mathcal{A})$, say $\|\cdot\|_b$, by

$$\|m\|_b := \|mb\| + \|bm\| \quad \text{for all } m \in \mathcal{M}(\mathcal{A}).$$

Clearly, $\|\cdot\|_b$ generates the strict topology on $\mathcal{M}(\mathcal{A})$. Note that $\|m\|_b \leq 2\|m\|$ for all $m \in \mathcal{M}(\mathcal{A})$.

Proposition 3.4. *Let \mathcal{A} be a σ -unital non-unital purely infinite simple C^* -algebra and let $A \in \mathcal{M}(\mathcal{A})$ be a positive element with $\|A\|_{ess} > 1$. Then A is a strict sum of projections.*

Proof. Every σ -unital, non-unital C^* -algebra of real rank zero has an approximate identity of projections; such an approximate identity can always be chosen to be countable and increasing, say $\{e_j\}$ ([28]).

Let $q_j = e_j - e_{j-1}$ setting $e_0 = 0$. Then $\sum_{j=1}^{\infty} q_j = I$, where the convergence is in the strict topology. Furthermore,

$$A = \sum_{j=1}^{\infty} A^{1/2} q_j A^{1/2}$$

where the convergence is also in the strict topology. By Lemma 3.2 (ii),

$$\left\| \sum_{j=n}^{\infty} A^{1/2} q_j A^{1/2} \right\| \geq \|A\|_{ess}$$

for every n . Thus the condition $\|A\|_{ess} > 1$ allows us to find a strictly increasing sequence of integers n_k starting with $n_0 = 1$ such that

$$\left\| \sum_{j=n_{k-1}}^{n_k-1} A^{1/2} q_j A^{1/2} \right\| > 1$$

for every k . Let

$$a_k := \sum_{j=n_{k-1}}^{n_k-1} A^{1/2} q_j A^{1/2}.$$

Then a_k is a positive element in \mathcal{A}^+ with $\|a_k\| > 1$ for every k and $A = \sum_{k=1}^{\infty} a_k$ in the strict topology. Thus

$$\left\| \sum_{k=n}^{\infty} a_k \right\|_b \rightarrow 0.$$

Apply Lemma 2.5 to a_1 to obtain a finite sum of projections $d_1 \in \mathcal{A}$, $d_1 \leq a_1$ with

$$\|d_1 - a_1\| < \frac{1}{2} \left\| \sum_{k=2}^{\infty} a_k \right\|_b.$$

Let $b_1 := a_1 - d_1 \in \mathcal{A}^+$ and hence $\|b_1\|_b \leq \|\sum_{k=2}^{\infty} a_k\|_b$. Then $A - d_1 = b_1 + \sum_{k=2}^{\infty} a_k$, and hence,

$$\|A - d_1\|_b \leq \|b_1\|_b + \|\sum_{k=2}^{\infty} a_k\|_b \leq 2\|\sum_{k=2}^{\infty} a_k\|_b.$$

Next, since $b_1 + a_2 \in \mathcal{A}^+$ and $\|b_1 + a_2\| \geq \|a_2\| > 1$, we can apply Lemma 2.5 to $b_1 + a_2$ to obtain a finite sum of projections $d_2 \leq b_1 + a_2$ with

$$\|b_1 + a_2 - d_2\| \leq \frac{1}{2}\|\sum_{k=3}^{\infty} a_k\|_b.$$

Thus, iterating, we can find for each k a finite sum d_k of projections in \mathcal{A} so that

$$\|A - \sum_{k=1}^n d_k\| \leq 2\|\sum_{k=n+1}^{\infty} a_k\|_b \rightarrow 0.$$

This proves that the sum $\sum_{k=1}^{\infty} d_k$ converges to A in the strict topology, and hence that A is a strict sum of projections, as claimed. \square

Remark 3.5. *In the course of the above proof we have proven that if \mathcal{A} is a σ -unital non-unital purely infinite simple C^* -algebra, and $A = \sum_{k=1}^{\infty} a_k$ in the strict topology, where $a_k \in \mathcal{A}^+$ and $\|a_k\| > 1$ for all k , then A is a strict sum of projections. The condition "purely infinite and simple" is the key assumption for the conclusion to hold in the eyes of key Lemma 2.5.*

4. The case $\|A\|_{ess} = 1$ and $\|A\| > 1$

The objective of this section is to prove that $\|A\|_{ess} = 1$ and $\|A\| > 1$ suffice to have A written as a strictly converging sum of projections in \mathcal{A} . We start with some technical preparations.

Lemma 4.1. *Let \mathcal{A} be any C^* -algebra of real rank zero and A be a positive element in $\mathcal{M}(\mathcal{A})$ such that $\|A\|_{ess} = 1$ and $\|A\| > 1$. Then there exist a positive element $A' \in \mathcal{M}(\mathcal{A})$, a real number $\lambda > 1$, and a nonzero projection $p \in \mathcal{A}$ such that*

- (i) $\|A'\|_{ess} = 1$,
- (ii) $A'p = pA' = 0$,
- (iii) $A' + \lambda p \leq A$.

Proof. Let $\delta = \|A\| - 1$. Define two positive continuous functions $h_1(t)$ and $h_2(t)$ on $[0, \|A\|]$ as follows:

$$h_1(t) := \begin{cases} 0 & t \in [0, 1 + \frac{\delta}{2}] \\ \text{linear} & t \in [1 + \frac{\delta}{2}, 1 + \frac{3\delta}{4}] \\ t & t \in [1 + \frac{3\delta}{4}, \|A\|] \end{cases} \quad \text{and} \quad h_2(t) := \begin{cases} t & t \in [0, 1 + \frac{\delta}{4}] \\ \text{linear} & t \in [1 + \frac{\delta}{4}, 1 + \frac{\delta}{2}] \\ 0 & t \in [1 + \frac{\delta}{2}, \|A\|] \end{cases}$$

Clearly, $h_1(t) + h_2(t) \leq t$ and $h_1(t)h_2(t) = 0$ for all t , hence, $h_1(A) + h_2(A) \leq A$ and $h_1(A)h_2(A) = 0$. Let π be the quotient map from $\mathcal{M}(\mathcal{A})$ to the corona algebra $\mathcal{M}(\mathcal{A})/\mathcal{A}$. Reasoning as in Lemma 3.2, $h_1(A) \in \mathcal{A}$ and $\|h_2(A)\|_{ess} = \|A\|_{ess} = 1$. Applying Lemma 2.3, approximate $h_1(A)$ by a positive element of finite spectrum satisfying

$$\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_m p_m \leq h_1(A)$$

where p_i are pairwise orthogonal nonzero projections in \mathcal{A} . For a sufficient approximation, $\alpha_i > 1$ holds for at least one i_0 . Set $\lambda := \alpha_{i_0}$, $p := p_{i_0}$, and $A' = h_2(A)$.

Then (i) is satisfied. Since $\lambda p \leq h_1(A)$ and hence $A' + \lambda p \leq A$, i.e., (iii) is satisfied. Since $h_1(A)A' = A'h_1(A) = 0$, it follows that $A'p = pA' = 0$. i.e., (ii) is satisfied. \square

The content of the following lemma can be found in the proof of Theorem 2.2 of [25].

Lemma 4.2. *Let \mathcal{A} be a σ -unital, non-unital C^* -algebra that has an approximate identity of projections. If $A \in \mathcal{M}(\mathcal{A})$ is a positive element, then for every $\epsilon > 0$ there exist three positive elements $A_1, A_2, A_3 \in \mathcal{M}(\mathcal{A})$ and there is a self-adjoint element $a \in \mathcal{A}$ with $\|a\| < \epsilon$ such that*

$$A = A_1 + A_2 + A_3 + a,$$

where all A_1, A_2, A_3 are in block-diagonal forms (see the detailed descriptions in the following proof).

Proof. The details were given in the proof of [25, 2.2], but we sketch them here for the convenience of the readers. Let $\{e_j\}$ be an approximate unit of projections. For $i \geq 1$ we will view $(e_i - e_{i-1})A^{\frac{1}{2}}(e_j - e_{j-1})$ as the (i, j) -entry of $A^{\frac{1}{2}}$, and view $(e_{n_{i+1}} - e_{n_i})A^{\frac{1}{2}}(e_{n_j} - e_{n_{j-1}})$ as the (i, j) -block entry of $A^{\frac{1}{2}}$.

Using a standard argument recursively on $A^{\frac{1}{2}}$ one can find an increasing sequence of indices $\{n_i\}$ starting with $n_0 = 0$ such that $A^{\frac{1}{2}}$ can be rewritten as a sum $A^{\frac{1}{2}} =: X + a$ of two self-adjoint elements, where setting $e_{n_0} = 0$,

$$a =: \sum_{i=1}^{\infty} \{ (e_{n_i} - e_{n_{i-1}})A^{\frac{1}{2}}(1 - e_{n_{i+1}}) + (1 - e_{n_{i+1}})A^{\frac{1}{2}}(e_{n_i} - e_{n_{i-1}}) \}$$

satisfies $\|a\| \leq \frac{\epsilon}{2\sqrt{\|A\|}+1}$ and $X =: A^{\frac{1}{2}} - a$. Then

$$\begin{aligned} X &= \sum_{i=1}^{\infty} (e_{n_{i+1}} - e_{n_i})A^{\frac{1}{2}}(e_{n_i} - e_{n_{i-1}}) \\ &\quad + \sum_{i=1}^{\infty} (e_{n_i} - e_{n_{i-1}})A^{\frac{1}{2}}(e_{n_i} - e_{n_{i-1}}) \\ &\quad + \sum_{i=1}^{\infty} (e_{n_i} - e_{n_{i-1}})A^{\frac{1}{2}}(e_{n_{i+1}} - e_{n_i}), \end{aligned}$$

the second sum above can be viewed as the main block diagonal, the first sum (resp., last sum) can be viewed as the first block diagonal below (resp., above) the main one. In this way, X is said to have a tri-block diagonal form.

Define

$$\begin{aligned} A_1 &=: X \sum_{i=1}^{\infty} (e_{n_{3i-2}} - e_{n_{3i-3}})X, \\ A_2 &=: X \sum_{i=1}^{\infty} (e_{n_{3i-1}} - e_{n_{3i-2}})X, \\ A_3 &=: X \sum_{i=1}^{\infty} (e_{n_{3i}} - e_{n_{3i-1}})X. \end{aligned}$$

Clearly, $A_1 + A_2 + A_3 = X^2 = A - A^{\frac{1}{2}}a_0 - a_0A^{\frac{1}{2}} + a_0^2$ and all three sums A_1 , A_2 , and A_3 strictly converge to positive elements of $\mathcal{M}(\mathcal{A})$. Set $a = A^{\frac{1}{2}}a_0 + a_0A^{\frac{1}{2}} - a_0^2$. Then $\|a\| < \epsilon$.

Via multiplication one sees that A_1 is of block diagonal with respect to the decomposition of the identity (of $\mathcal{M}(\mathcal{A})$)

$$1 = e_{n_3} \oplus (e_{n_6} - e_{n_3}) \oplus \cdots \oplus (e_{n_{3i}} - e_{n_{3i-3}}) \oplus \cdots,$$

A_2 is of block diagonal with respect to the decomposition

$$1 = (e_{n_4} - e_{n_1}) \oplus (e_{n_7} - e_{n_4}) \oplus \cdots \oplus (e_{n_{3i+1}} - e_{n_{3i-2}}) \oplus \cdots,$$

and A_3 is of block diagonal with respect to the decomposition

$$1 = (e_{n_5} - e_{n_2}) \oplus (e_{n_8} - e_{n_5}) \oplus \cdots \oplus (e_{n_{3i+2}} - e_{n_{3i-1}}) \oplus \cdots.$$

□

Lemma 4.3. *Let \mathcal{A} be a σ -unital, non-unital C^* -algebra of real rank zero and let $A \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$ be a positive element. Then for every $\epsilon > 0$ there exist a sequence $\{q_k\}_{k=1}^\infty$ of pairwise orthogonal nonzero projections in \mathcal{A} , a bounded sequence $\{\lambda_k\}_{k=1}^\infty$ of positive real numbers, a positive element $A_0 \in \mathcal{M}(\mathcal{A})$, and a self-adjoint element $a \in \mathcal{A}$ such that the following hold:*

- (i) $\|a\| < \epsilon$.
- (ii) $\sum_{k=1}^\infty q_k$ converges in the strict topology of $\mathcal{M}(\mathcal{A})$.
- (iii) $A = A_0 + \sum_{k=1}^\infty \lambda_k q_k + a$.
- (iv) $\lim_{k \rightarrow \infty} \lambda_k = \|A\|_{ess}$.

Proof. Applying Lemma 4.2, one has a decomposition

$$A = A_1 + A_2 + A_3 + a$$

where $a \in \mathcal{A}$ is self-adjoint, $\|a\| < \epsilon$, $A_1, A_2, A_3 \in \mathcal{M}(\mathcal{A})$ are positive element in block-diagonal forms, as described in the proof of 4.2. Let $a_{i,j}$ be the j th-block on the diagonal of A_i for $i = 1, 2, 3$ and $j = 1, 2, \dots$. All $a_{i,j}$ are positive elements in \mathcal{A} and $\sum_{i=1}^3 \sum_{j=1}^\infty a_{i,j} = A_1 + A_2 + A_3$ converges in the strict topology.

We construct by induction a sequence of positive numbers λ_k and mutually orthogonal projections $q_k \in \mathcal{A}$ such that for each k

$$\begin{aligned} \|A\|_{ess} &\geq \lambda_k > \|A\|_{ess} - 1/2^k \\ \sum_{j=1}^k \lambda_k q_k &\leq \sum_{i=1}^3 \sum_{j=1}^{n_k} a_{i,j}, \end{aligned}$$

where $\{n_k\}$ is an increasing sequence of natural numbers.

For $k = 1$, since

$$\left\| \sum_{i=1}^3 \sum_{j=1}^n a_{i,j} \right\| \uparrow \|A - a\| \geq \|A - a\|_{ess} = \|A\|_{ess},$$

we can choose an integer n_1 such that $\left\| \sum_{i=1}^3 \sum_{j=1}^{n_1} a_{i,j} \right\| > \|A\|_{ess} - 1/2$. By Lemma 2.3 applied to $\sum_{i=1}^3 \sum_{j=1}^{n_1} a_{i,j} \in \mathcal{A}$ one can find an approximation of $\sum_{i=1}^3 \sum_{j=1}^{n_1} a_{i,j}$ by a positive element of finite spectrum belonging to \mathcal{A} , $\sum_{i=1}^m \alpha_i p_i$, with $\alpha_i > 0$, p_i

mutually orthogonal nonzero projections of \mathcal{A} , and

$$\begin{aligned} \sum_{i=1}^m \alpha_i p_i &\leq \sum_{i=1}^3 \sum_{j=1}^{n_1} a_{i,j}, \\ \left\| \sum_{i=1}^3 \sum_{j=1}^{n_1} a_{i,j} - \sum_{i=1}^m \alpha_i p_i \right\| &\leq \frac{1}{2} (\|A\|_{ess} - 1/2). \end{aligned}$$

Then at least one of $\alpha_1, \alpha_2, \dots, \alpha_m$, say α_j , satisfies

$$\alpha_j > \|A\|_{ess} - 1/2.$$

Let $\lambda_1 =: \min\{\alpha_j, \|A\|_{ess}\}$ and $q_1 =: p_j$, where q_1 is a nonzero projection from \mathcal{A} . Then one has the desired inequality:

$$\lambda_1 q_1 \leq \sum_{i=1}^3 \sum_{j=1}^{n_1} a_{i,j}.$$

For $k = 2$ take an integer $m \geq n_1 + 3$ so that the sum $\sum_{i=1}^3 \sum_{j=m+1}^{\infty} a_{i,j}$ is orthogonal to $\sum_{i=1}^3 \sum_{j=1}^{n_1} a_{i,j}$ in the sense

$$\left(\sum_{i=1}^3 \sum_{j=m+1}^{\infty} a_{i,j} \right) \left(\sum_{i=1}^3 \sum_{j=1}^{n_1} a_{i,j} \right) = \left(\sum_{i=1}^3 \sum_{j=1}^{n_1} a_{i,j} \right) \left(\sum_{i=1}^3 \sum_{j=m+1}^{\infty} a_{i,j} \right) = 0.$$

Such an m exists by the construction of A_1, A_2, A_3 .

Since $\left\| \sum_{i=1}^3 \sum_{j=m+1}^{\infty} a_{i,j} \right\|$ increases to $\|A - a - \sum_{i=1}^3 \sum_{j=1}^m a_{i,j}\|$ and

$$\|A - a - \sum_{i=1}^3 \sum_{j=1}^m a_{i,j}\| \geq \|A - a - \sum_{i=1}^3 \sum_{j=1}^m a_{i,j}\|_{ess} = \|A\|_{ess},$$

repeating the above argument for $k = 1$, choose an integer $n_2 > m$ such that

$$\left\| \sum_{i=1}^3 \sum_{j=m+1}^{n_2} a_{i,j} \right\| > \|A\|_{ess} - 1/2^2.$$

As for the case $k = 1$, one can choose a nonzero projection q_2 in \mathcal{A} and $\lambda_2 > 0$ such that

$$\|A\|_{ess} \geq \lambda_2 > \|A\|_{ess} - 1/2^2,$$

$$\lambda_2 q_2 \leq \sum_{i=1}^3 \sum_{j=m+1}^{n_2} a_{i,j}.$$

Clearly, $q_2 q_1 = 0$ and the obvious inequality

$$\sum_{i=1}^3 \sum_{j=m+1}^{n_2} a_{i,j} \leq \sum_{i=1}^3 \sum_{j=n_1+1}^{n_2} a_{i,j}$$

guarantees that

$$\lambda_1 q_1 + \lambda_2 q_2 \leq \sum_{i=1}^3 \sum_{j=1}^{n_2} a_{i,j}.$$

Proceeding recursively, one constructs a sequence of pairwise orthogonal projections $\{q_k\}$ and a sequence of positive numbers $\{\lambda_k\}$ with the required properties.

It is now routine to prove that $\sum_{k=1}^{\infty} \lambda_k q_k$ and hence also $\sum_{k=1}^{\infty} q_k$ converge in the strict topology of $\mathcal{M}(\mathcal{A})$. Then setting

$$A_o = A_1 + A_2 + A_3 - \sum_{k=1}^{\infty} \lambda_k q_k \in \mathcal{M}(\mathcal{A})$$

satisfies (i)–(iv). \square

We now reach our second key lemma.

Lemma 4.4. *Let \mathcal{A} be a σ -unital non-unital purely infinite simple C^* -algebra and A be a positive element of $\mathcal{M}(\mathcal{A})$ such that $\|A\|_{ess} = 1$ and $\|A\| > 1$. Then there exists a sequence $\{f_k\}_{k=1}^{\infty}$ of pairwise orthogonal projections in \mathcal{A} such that*

- (i) $\sum_{k=1}^{\infty} f_k$ converges in the strict topology in $\mathcal{M}(\mathcal{A})$, and
- (ii) $\|f_k A f_k\| > 1$ for all k .

Proof. By Lemma 4.1 there is a positive element A' of $\mathcal{M}(\mathcal{A})$ with $\|A'\|_{ess} = 1$, a projection q_0 of \mathcal{A} with $q_0 A' = A' q_0 = 0$, and a scalar $\lambda_0 > 1$ such that

$$A_0 := A - A' - \lambda_0 q_0 \geq 0.$$

By Lemma 4.3 applied to A' and $\epsilon = 1$, there is a sequence of positive real numbers $\lambda'_k \rightarrow 1$, a sequence of pairwise orthogonal nonzero projections $\{q'_k\}_{k=1}^{\infty} \in \mathcal{A}$, a self adjoint element $a = a^* \in \mathcal{A}$, and a positive element $A'_0 \in \mathcal{M}(\mathcal{A})$ such that

$$A' = A'_0 + \sum_{k=1}^{\infty} \lambda'_k q'_k + a.$$

Notice that we can choose $q_0 q'_k = q'_k q_0 = 0$ for all k because $q_0 A' = A' q_0 = 0$ (just replace \mathcal{A} with $(1 - q_0)\mathcal{A}(1 - q_0)$ when applying Lemma 4.3).

Choose a subsequence $\lambda_k := \lambda'_{n_k}$ such that

$$(1) \quad |\lambda_k - 1| < (\lambda_0 - 1)/4^{k+1}.$$

Since $\sum_{k \geq 1} q'_{n_k}$ still converges in the strict topology of $\mathcal{M}(\mathcal{A})$, by passing if necessary to a subsequence, one can further assume that

$$(2) \quad \max\{\|(a_+)^{1/2} q'_{n_k}\|, \|(a_-)^{1/2} q'_{n_k}\|\} < \frac{\sqrt{\lambda_0 - 1}(\sqrt{2} - 1)}{2^{k+1}\sqrt{2}},$$

where a_- and a_+ denote the negative and positive parts of a , respectively. Since \mathcal{A} is simple and purely infinite, for each $k \geq 1$ there exists a subprojection q_k of q'_{n_k} such that $q_k \sim q_0$.

Set

$$q := \sum_{k=0}^{\infty} q_k$$

$$A_{00} = A_0 + A'_0 + \sum_{i \notin \{n_k\}} \lambda_i q'_i + \sum_{k=1}^{\infty} \lambda'_{n_k} (q'_{n_k} - q_k).$$

Then q is a projection of $\mathcal{M}(\mathcal{A})$ (as a direct sum of countably many mutually orthogonal copies of q_0), A_{00} is a positive element of $\mathcal{M}(\mathcal{A})$, and

$$(3) \quad A = A_{00} + \sum_{k=0}^{\infty} \lambda_k q_k + a.$$

Let $\{e_i\}_{i=0}^\infty$ be the sequence of mutually orthogonal rank-one projections in $\mathbb{B}(\ell_2)$ corresponding to the standard basis of ℓ_2 and let ρ be a unital (isometrical) $*$ -embedding $\mathbb{B}(\ell_2) \rightarrow q\mathcal{M}(\mathcal{A})q$ for which

$$\rho(e_i) = q_i \quad \text{for all } i \geq 0.$$

It is easy to verify that the following matrix u is unitary

$$\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & \dots & & \\ (1/\sqrt{2})^2 & (1/\sqrt{2})^2 & -1/\sqrt{2} & 0 & \dots & \\ \dots & \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & \dots & \\ (1/\sqrt{2})^n & (1/\sqrt{2})^n & (1/\sqrt{2})^{n-1} & \dots & (1/\sqrt{2})^3 & (1/\sqrt{2})^2 & -1/\sqrt{2} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

and thus $\rho(u)$ is a unitary element of $q\mathcal{M}(\mathcal{A})q$. Define for all $k \geq 0$

$$f_k =: \rho(u^* e_k u).$$

Then $\{f_k\}_{k=0}^\infty$ is a sequence of pairwise orthogonal (equivalent) projections in \mathcal{A} and

$$\sum_{k=0}^\infty f_k = \rho(u) \left(\sum_{k=0}^\infty q_k \right) \rho(u)^* = q.$$

Since $A_{00} \geq 0$ it follows from (3) that

$$\begin{aligned} \|f_k A f_k\| &\geq \|f_k \left(\sum_{j=0}^\infty \lambda_j q_j \right) f_k\| - \|f_k a f_k\| \\ &= \|\rho(u^* e_k u \left(\sum_{j=0}^\infty \lambda_j e_j \right) u^* e_k u)\| - \|f_k a f_k\| \\ &= \|e_k u \left(\sum_{j=0}^\infty \lambda_j e_j \right) u^* e_k\| - \|f_k a f_k\| \\ &= \left(u \left(\sum_{j=0}^\infty \lambda_j e_j \right) u^* \right)_{k,k} - \|f_k a f_k\|. \end{aligned}$$

Claim 1: $\left(u \left(\sum_{j=0}^\infty \lambda_j e_j \right) u^* \right)_{k,k} > 1 + (\lambda_0 - 1)/2^{k+2}$ for all k .

For ease of computations, notice that

$$(4) \quad u_{i,j} = \begin{cases} (1/\sqrt{2})^{i+1} & j = 0 \\ (1/\sqrt{2})^{i+2-j} & 1 \leq j \leq i \\ -1/\sqrt{2} & j = i+1 \\ 0 & j > i+1 \end{cases}$$

and thus for all $i, j \geq 0$,

$$(5) \quad (u e_0 u^*)_{i,i} = |u_{i,0}|^2 = 1/2^{i+1}$$

$$(6) \quad |u_{i,j}| \leq (1/\sqrt{2})^{i+1-j}$$

Then

$$\begin{aligned}
\left(u\left(\sum_{j=0}^{\infty}\lambda_j e_j\right)u^*\right)_{k,k} &= \left(u\left(I+(\lambda_0-1)e_0+\sum_{j=1}^{\infty}(\lambda_j-1)e_j\right)u^*\right)_{k,k} \\
&= 1+(\lambda_0-1)/2^{k+1}-\left|\sum_{i=1}^{k+1}u_{k,i}(\lambda_i-1)\overline{u_{k,i}}\right| && \text{(by (4))} \\
&\geq 1+(\lambda_0-1)/2^{k+1}-\sum_{i=1}^{k+1}|u_{k,i}|^2|\lambda_i-1| \\
&> 1+(\lambda_0-1)/2^{k+1}-\sum_{i=1}^{k+1}(1/2)^{k+1-i}(\lambda_0-1)/4^{i+1} && \text{(by (1) and (6))} \\
&> 1+(\lambda_0-1)/2^{k+1}-\frac{(\lambda_0-1)}{2^{k+3}} \\
&> 1+(\lambda_0-1)/2^{k+2},
\end{aligned}$$

which proves Claim 1.

Claim 2: $\|f_k a f_k\| < (\lambda_0 - 1)/2^{k+2}$ for all k .

Indeed,

$$\begin{aligned}
\|f_k a f_k\| &= \|\rho(u^* e_k u) a \rho(u^* e_k u)\| \\
&= \|q_k \rho(u) a \rho(u)^* q_k\| && \text{(since } \rho(u) \text{ is unitary)} \\
&= \|q_k \rho(u) \left(\sum_{i=0}^{\infty} q_i\right) a \left(\sum_{j=0}^{\infty} q_j\right) \rho(u^*) q_k\| && \text{(since } \rho(u) = q \rho(u) = \rho(u) q) \\
&= \left\| \sum_{i,j=0}^{k+1} q_k \rho(u) q_i a q_j \rho(u^*) q_k \right\| && \text{(by 4)} \\
&\leq \sum_{i,j=0}^{k+1} \|q_k \rho(u) q_i\| \|q_j \rho(u^*) q_k\| \|q_i a q_j\| \\
&= \sum_{i,j=0}^{k+1} |u_{k,i}| |u_{k,j}| \|q_i a q_j\| \\
&\leq \sum_{i,j=0}^{k+1} |u_{k,i}| |u_{k,j}| (\|q_i a_+ q_j\| + \|q_i a_- q_j\|) && \text{(since } a = a_+ - a_-) \\
&\leq \sum_{i,j=0}^{k+1} (1/\sqrt{2})^{k+1-i} (1/\sqrt{2})^{k+1-j} \frac{(\lambda_0-1)(\sqrt{2}-1)^2}{2^{i+j+2}} && \text{(by (6) and (2))} \\
&= \frac{(\lambda_0-1)(\sqrt{2}-1)^2}{2^{k+3}} \sum_{i=0}^{k+1} (1/\sqrt{2})^i \sum_{j=0}^{k+1} (1/\sqrt{2})^j \\
&= \frac{(\lambda_0-1)(\sqrt{2}-1)^2}{2^{k+3}} \frac{2}{(\sqrt{2}-1)^2} \left(1 - \left(\frac{1}{\sqrt{2}}\right)^{k+2}\right)^2 \\
&< (\lambda_0-1)/2^{k+2},
\end{aligned}$$

which proves Claim 2.

From (4), Claim 1 and Claim 2, we see that for all k ,

$$\|f_k A f_k\| > 1 + (\lambda_0 - 1)/2^{k+2} - (\lambda_0 - 1)/2^{k+2} = 1.$$

□

Remark 4.5. *The key idea in the proof for Lemma 4.4 above is that by acting within a copy of $\mathbb{B}(l_2)$ that is identified with the corner $q\mathcal{M}(\mathcal{A})q$ of $\mathcal{M}(\mathcal{A})$, the unitary matrix u permits to turn the diagonal operator $\sum_{k=0}^{\infty} \lambda_k q_k$ which has one entry larger than 1 and all the other entries “close” to 1 into an operator with all the diagonal entries strictly larger than 1. This “spreading out” action of u can be illustrated directly in $\mathbb{B}(l_2)$ by showing that if for some $t > 0$ we set $D := I + te_o \in \mathbb{B}(l_2)$, i.e., the diagonal operator with diagonal sequence*

$$\langle 1 + t, 1, 1, 1, \dots \rangle,$$

then the diagonal sequence of uDu^ is*

$$\langle 1 + 1/2t, 1 + 1/2^2t, \dots, 1 + 1/2^nt, \dots \rangle$$

where indeed each diagonal entry is larger than 1.

Proposition 4.6. *Let \mathcal{A} be a σ -unital non-unital purely infinite simple C^* -algebra and let $A \in \mathcal{M}(\mathcal{A})^+$ with $\|A\|_{ess} = 1$ and $\|A\| > 1$. Then A is a strict sum of projections.*

Proof. Since $\|A\|_{ess} = 1$ and $\|A\| > 1$, there exists a sequence $\{f_k\}_{k=0}^{\infty}$ of pairwise orthogonal projections in \mathcal{A} satisfying all conditions of Lemma 4.4.

Let $p = \sum_{k=0}^{\infty} f_k$. Then p is a projection of $\mathcal{M}(\mathcal{A})$. Rewrite A as

$$A = A^{1/2}(1-p)A^{1/2} + \sum_{k=0}^{\infty} A^{1/2}f_kA^{1/2}$$

where the sum converges in the strict topology in $\mathcal{M}(\mathcal{A})$ and $\|A^{1/2}f_kA^{1/2}\| > 1$ for all $k \geq 0$ by condition (iii) of Lemma 4.4. Now choose a sequence of mutually orthogonal projections g_k of \mathcal{A} whose sum converges to the identity. Then $A^{1/2}(1-p)g_k(1-p)A^{1/2} \in \mathcal{A}^+$ for every k and

$$A^{1/2}(1-p)A^{1/2} = \sum_{k=1}^{\infty} A^{1/2}(1-p)g_k(1-p)A^{1/2}.$$

in the strict topology. Let

$$a_k := A^{1/2}(1-p)g_k(1-p)A^{1/2} + A^{1/2}f_kA^{1/2}.$$

Then $a_k \in \mathcal{A}^+$, $\|a_k\| \geq \|A^{1/2}f_kA^{1/2}\| > 1$ for all k and

$$A = \sum_{k=1}^{\infty} a_k$$

in the strict topology. By Remark 3.5, A is a strict sum of projections. □

This provides the last substantial step in the proof of our main theorem.

Proof of Theorem 1.1. First, the necessity. If A is a strict sum of projections belonging to \mathcal{A} , then either the number of projections is finite, in which case $A \in \mathcal{A}$ (case (iv)), or it is infinite, in which case $\|A\|_{ess} \geq 1$ by Lemma 3.2. It is clear that $\|A\| \geq 1$. If $\|A\| = 1$, then A must be itself a projection (case (iii)). Indeed it is

well known that if p, q are two projections and $\|p + q\| = 1$ then p and q must be orthogonal. Finally, if $\|A\| > 1$ then we can have either $\|A\|_{ess} > 1$ (case (i)) or $\|A\|_{ess} = 1$ (case(ii).)

Now the sufficiency.

- (i) Proposition 3.4
- (ii) Proposition 4.6
- (iii) If $A = p \in \mathcal{M}(\mathcal{A})$ is a projections, then pAp has an increasing approximate identity of projections ([28]), say f_n , and hence, $A = p = \sum_{n=1}^{\infty} (f_n - f_{n-1})$ as a strict sum of projections of \mathcal{A} (where f_0).
- (iv) Nothing to prove. □

Remark 4.7. *We would like to point out that due to [12, Theorem 1.1], the operator $A = I + (1/2)e \in \mathbb{B}(l_2)$, where e is a rank one projection, cannot be written as a strongly convergent sum of projections in $\mathbb{B}(l_2)$. However, if we unitarily embed $\mathbb{B}(l_2)$ in the multiplier algebra $\mathcal{M}(\mathcal{A})$ where \mathcal{A} is a σ -unital, nonunital purely infinite simple C^* -algebra, then A can be written as a strictly convergent sum of projections in \mathcal{A} . This is due to the much richer structure of $\mathcal{M}(\mathcal{A})$ than $\mathbb{B}(l_2)$.*

REFERENCES

- [1] J. Antezana, P. Massey, M. Ruiz and D. Stojanoff, *The Schur-Horn Theorem for operators and frames with prescribed norms and frame operator* Illinois J. of Math., (2007) Preprint
- [2] B. Blackadar, *K-theory for operator algebras*, (1986) Springer-Verlag, New York
- [3] L. Brown and G. Pedersen, *C^* -algebras of real rank zero* J. Funct. Anal., **99** (1991) 131-149
- [4] K. Davidson, *C^* -algebras by example* Fields Institute Monographs, **6** (1996) American Mathematical Society, Providence, RI
- [5] K. Dykema, D. Freeman, K. Kornelson, D. Larson, M. Ordower and E. Weber, *Ellipsoidal tight frames and projection decompositions of operators* Illinois J. Math., **48** (2004) no. 2, 477-489
- [6] G. A. Elliott, *Derivations of matroid C^* -algebras. II.* Ann. of Math. (2), **100** (1974) 407-422
- [7] G. A. Elliott and G. Gong, *On the classification of C^* -algebras of real rank zero. II* Ann. of Math. (2), **144** (1996) no. 3, 497-610
- [8] G. Elliott and M. Rordam, *Perturbation of Hausdorff moment sequences, and an application to the theory of C^* -algebras of real rank zero* Operator Algebras, The Abel Symposium 2004, (2006) Springer Verlag, 97-115
- [9] P. Fillmore, *On sums of projections* J. Funct. Anal., **4** (1969) 146-152
- [10] M. Frank and D. Larson, *A module frame concept for Hilbert C^* -modules* Contemp. Math., **247** (1999) 207-233
- [11] M. Frank and D. Larson, *Frames in Hilbert C^* -modules and C^* -algebras* J. Operator Theory, **48** (2002) no. 2, 273-314
- [12] V. Kaftal and P. W. Ng and S. Zhang, *Strong sums of projections in $\mathbb{B}(\mathcal{H})$ and in von Neumann factors* J. Funct. Anal., **257** (2009) 2497-2529
- [13] K. Kornelson and D. Larson, *Rank-one decomposition of operators and construction of frames* Contemp. Math., **345** (2004) 203-214
- [14] H. Lin, *Skeleton C^* -algebras* Canad. J. Math., **44** (1992) 324-341
- [15] H. Lin, *Generalized Weyl-von Neumann theorems. II.* Math. Scand, **77** (1995) no. 1, 129-147
- [16] H. Lin, *Embedding an AH-algebra into a simple C^* -algebra with prescribed KK -data* K-Theory, **24** (2001) no. 2, 135-156
- [17] H. Lin and S. Zhang, *On infinite simple C^* -algebras* J. Funct. Anal., **100** (1991) no. 1, 221-231

- [18] H. Lin, *An introduction to the classification of amenable C^* -algebras* World Scientific Publishing Co., Inc., (2001) River Edge, NJ
- [19] F. Perera and M. Rordam, *AF-embeddings into C^* -algebras of real rank zero* J. Funct. Anal., **217** (2004) no. 1, 142-170
- [20] N.E. Wegge-Olsen, *K -Theory and C^* -Algebras* Oxford University Press, Oxford, (1993)
- [21] S. Zhang, *On the structure of projections and ideals of corona algebras* Canad. J. Math., **41** (1989) no. 4, 721-742
- [22] S. Zhang, *A property of purely infinite simple C^* -algebras* Proc. Amer. Math. Soc., **109** (1990) no. 3, 717-720
- [23] S. Zhang, *C^* -algebras with real rank zero and the internal structure of their corona and multiplier algebras. III* Canad. J. Math., **42** (1990) no. 1, 159-190
- [24] S. Zhang, *Diagonalizing projections in multiplier algebras and in matrices over a C^* -algebra* Pacific J. Math, **145** (1990) no. 1, 181-200
- [25] S. Zhang, *A Riesz decomposition property and ideal structure of multiplier algebras* J. Operator Theory, **24** (1990) no. 2, 209-225
- [26] S. Zhang, *Matricial structure and homotopy type of simple C^* -algebras with real rank zero* J. Operator Theory, **26** (1991) 283-312
- [27] S. Zhang, *Ideals of generalized Calkin algebras* Contemp. Math., **120** (1991) 193-198
- [28] S. Zhang, *K_1 -groups, quasidiagonality, and interpolation by multiplier projections* Trans. Amer. Math. Soc., **325** (1991) no. 2, 793-818
- [29] S. Zhang, *Certain C^* -algebras with real rank zero and their corona and multiplier algebras. I.* Pacific J. Math., **155** (1992) no. 1, 169-197
- [30] S. Zhang, *C^* -algebras with real rank zero and their corona and multiplier algebras. IV* Internat. J. Math., **3** (1992) no. 2, 309-330
- [31] S. Zhang, *Certain C^* -algebras with real rank zero and their corona and multiplier algebras. I* Pacific J. Math., **155** (1992) no. 1, 169-197
- [32] S. Zhang, *Certain C^* -algebras with real rank zero and their corona and multiplier algebras. II K -theory,* **6** (1992) no. 1, 1-27
- [33] S. Zhang, *Quasidiagonalizing unitaries and the generalized Weyl-von Neumann theorem* Algebraic methods in operator theory, (1994) Birkhauser, Boston, MA, USA

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CINCINNATI, P. O. BOX 210025, CINCINNATI, OH, 45221-0025, USA

E-mail address: victor.kaftal@UC.Edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LOUISIANA, 217 MAXIM D. DOUCET HALL, P.O. BOX 41010, LAFAYETTE, LOUISIANA, 70504-1010, USA

E-mail address: png@louisiana.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CINCINNATI, P.O. BOX 210025, CINCINNATI, OH, 45221-0025, USA

E-mail address: zhangs@email.uc.edu